Harmonic Motion (Sections 4.4, 4.7)

Consider the vibrating spring with a weight of mass m. At the equilibrium point, there are two forces acting on the mass.

- Force of gravity: $F_G = mg$
- Restoring Force: $R(x) = -kx$ where k is the *spring constant*

By Newton's second law of motion: $mg - kx_0 = 0$ or $mg = kx_0$
If the spring is moving, there is a third force acting on it.

- Damping Force: $D(v) = -\mu v$ where μ is a constant (This is also called the frictional force which is assumed to be proportional to the velocity)

Sometimes there is an outside force acting on the system.

- External Force: $f(t)$

By Newton's second law of motion, we have $mg - kx - \mu v + f(t) = ma$
Since $mg = kx_0$, $v = x'$, $a = x''$, we obtain $-k(x - x_0) - \mu x' + f(t) = mx''$
Let $y = x - x_0$, the equation can be written as

$$my'' + \mu y' + ky = f(t)$$

This is the second order linear equation with constant coefficients m, μ and k.
The *inhomogeneous term* on the right hand side is also called *forcing term* because it is the external force acting on the spring-mass system.

Unforced Harmonic Motion

If there is no outside force, the equation becomes a homogeneous equation:

$$my'' + \mu y' + ky = 0$$

Divide both sides by m, we obtain $y'' + \frac{\mu}{m}y' + \frac{k}{m} = 0$.
Let $c = \frac{\mu}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$, we have

$$y'' + 2cy' + \omega_0^2 y = 0$$

Note that c is called the *damping constant* and ω_0 is called the *natural frequency*.

Simple Harmonic Motion

If there is no damping (friction is ignored), $c = 0$ and the equation becomes $y'' + \omega_0^2 y = 0$

Its characteristic equation has two imaginary solutions: $\lambda = \pm i\omega_0$

Then the general solution to the equation is $y(t) = a\cos\omega_0 t + b\sin\omega_0 t$

We can put the solution in a 'better' form $y(t) = A\cos(\omega_0 t - \phi)$ where $A = \sqrt{a^2 + b^2}$ and $\tan\phi = \frac{b}{a}$

This solution describes the motion of the spring: It oscillates up and down freely with the amplitude A and the natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$.

The period is $T = \frac{2\pi}{\omega_0}$.

Damped Harmonic Motion

This time $c \neq 0$, and we need to solve the equation in general $y'' + 2cy' + \omega_0^2 y = 0$

Its characteristic equation is $\lambda^2 + 2c\lambda + \omega_0^2 = 0$, which has two solutions: $\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$ and $\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}$.

1. Overdamped Case ($c > \omega_0$): $y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

   In this case, there is no oscillation from one side to another of the equilibrium point.

2. Underdamped Case ($c < \omega_0$): $y(t) = Ae^{-ct}\cos\omega t + Be^{-ct}\sin\omega t$

   In this case, the cosine and sine terms cause the function to oscillate (with frequency ω) up and down with decreasing amplitude until it reaches the equilibrium.

3. Critically-damped case ($c = \omega_0$): $y(t) = Ae^{-ct} + Bte^{-ct}$

   This case is the dividing line that separates the two cases and the solution also fails to oscillate.

Examples

A weight is attachd to a spring and the system is allowed to come to rest.

a) If the mass is 0.1 kg, spring constant is 3.6 kg/s^2 and there is no damping but the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. Find the amplitude and the frequency of the resulting motion. Plot the solution and describe the motion of the system.

b) If the mass is 2 kg, spring constant 2 kg/s^2 and the damping constant is 1.25 kg/s. Find the position function and describe the motion of the system.

c) If the mass is 1 kg, spring constant 1 kg/s^2 and the damping constant is 1 kg/s. Find the position function and describe the motion of the system.

Homework: Section # 11, 13 and 16.